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On the String Actions for the Generalized Two-dimensional Yang-Mills Theories

Yuji Sugawara[1]

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606, Japan[2]

Abstract

We study the structures of partition functions of the large N generalized two-dimensional Yang-Mills theories (gYM_2) by recasting the higher Casimirs. We clarify the appropriate interpretations of them and try to extend the Cordes-Moore-Ramgoolam's topological string model describing the ordinary YM_2 [4] to those describing gYM_2. We present the expressions of the appropriate operators to reproduce the higher Casimir terms in gYM_2. The concept of "deformed gravitational descendants" will be introduced for this purpose.

[1]e-mail address: sugawara@yukawa.kyoto-u.ac.jp

[2]Present adress: Department of Physics, Osaka University Machikaneyama 1-1, Toyonaka, Osaka 560, Japan

1. It is an old problem to understand relationships between Yang-Mills theory and string theory [1], especially for the two-dimensional case (YM_2) [2]. In early years of 90's one great progress was given by Gross and Taylor within the framework of YM_2 on any compact Riemann surface M [3]. In these celebrated works they investigated in detail the large N expansion of the partition function by making use of some group theoretical techniques, and showed that it is realized as an asymptotic series of which terms are all some homotopy invariants of ramified covering maps onto the considered Riemann surface M. This strongly suggests the possibility of reformulating YM_2 as a theory of topological string with the space of maps $f : \Sigma \longrightarrow M$ as the configuration space.

Inspired with these Gross-Taylor's studies, Cordes, Moore and Ramgoolam gave an elegant implication [4]. They firstly considered the 0-area limit (or equivalently, the weak coupling limit) of the large N YM_2 and proved that each term of the Gross-Taylor's asymptotic series is exactly equal to the Euler number of the moduli space of branched covers. Based on this observation they constructed the world sheet action of a topological string model corresponding to the large N YM_2. Their topological string theory is formulated to calculate the Euler number of the moduli space and, to this aim, includes some extra degrees of freedom - the "co-fields" [4] (see also [10]). For the case of non-zero area, they gave a conjecture (and partially proved) that by adding the simple perturbation of the "area operator" one can reproduce the result of the non-zero area case. They also presented a stimulating speculation; the appropriate perturbations of the gravitational descendants of area operators might correspond to the "generalized 2-dim Yang-Mills theories" (gYM_2) [7, 8], which are given by replacing the 2nd Casimir with some higher Casimirs in the heat kernel Boltsmann weight of YM_2 [5, 6, 7].

In this article we shall present a detailed study of gYM_2 motivated with this speculation. We shall exhibit the world-sheet actions for the topological string models describing gYM_2, in other words, construct the suitable perturbation terms to the CMR's string model reproducing the higher Casimirs. To this aim we shall establish the rule to translate the algebraic data of higher Casimirs appearing in gYM_2 into some geometric data fitted for the topological string theory. Our suitable perturbation terms will be expressed by the *deformed* gravitational descendants, which will be defined later.

2. We shall start with a short review for the work [4]. Let M be an arbitrary compact Riemann surface with genus p, and G be a compact group (gauge group). The two-dimensional

pure Yang-Mills theory (YM_2) on M is usually defined by

$$S_{YM_2}(A_\mu) = \frac{1}{4g^2}\int_M dv\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}), \tag{1}$$

or by an equivalent form;

$$S_{YM_2}(A_\mu,\phi) = \frac{1}{2}\int_M i\mathrm{Tr}(\phi F) + \frac{g^2}{4}\int_M dv\,\mathrm{Tr}(\phi^2). \tag{2}$$

Here ϕ is a scalar field valued in the Lie algebra of G. Such a reformulation of YM_2 makes it clearer that YM_2 can be regarded as a topological gauge theory (the BF-theory) $S_{BF} = \frac{1}{2}\int_M i\mathrm{Tr}(\phi F)$ with the perturbation term $\frac{g^2}{4}\int_M dv\,\mathrm{Tr}(\phi^2)$ which breaks down the topological invariance. Some detailed investigations from this point of view are given in [7].

The partition function of YM_2 is calculable by the lattice method [5, 6] or by the continuum approach [7]. The exact result is summarized as the following famous formula;

$$Z_{YM_2}(M,G) = \sum_R (\dim R)^{2-2p}\, e^{-\frac{g^2}{2}AC_2(R)} \tag{3}$$

where the summation is taken over all the equivalence classes of irreducible representations of G and $C_2(R)$ is the 2nd Casimir operator.

Utilizing some group theoretical techniques Gross and Taylor derived the formula of $1/N$ asymptotic expansion of this partition function (3) for the case of $G = SU(N)$ with the redefinition of coupling constant $\lambda = g^2N$ [3]. After that Cordes, Moore and Ramgoolam made a refinement of the Gross-Taylor's formula [4]. This can be written in the following way only for the chiral sector[3];

$$\begin{aligned} Z^+_{YM_2}(M,SU(N)) = 1 + \sum_{n=1}^{\infty}\sum_{B=0}^{\infty}\left(\frac{1}{N}\right)^{n(2p-2)+B} & e^{-\frac{1}{2}n\lambda A}\, e^{\frac{n^2}{2N^2}\lambda A}\sum_{k=0}^{\infty}\frac{1}{k!}(-\lambda A)^k \sum_{L=0}^{\infty}\chi(\mathcal{C}_L(M)) \\ \times & \sum_{v_1,\ldots,v_L\in S_n\backslash\{1\}}\;\sum_{p_1,\ldots,p_k\in T_2}\;\sum_{s_1,t_1,\ldots,s_p,t_p\in S_n} \\ & \times\frac{1}{n!}\delta(v_1\cdots v_L\,p_1\cdots p_k\prod_{i=1}^{p}s_it_is_i^{-1}t_i^{-1})\,\delta_{\sum_{i=1}^{L}(n-K_{v_i})+k\ ,B}\quad . \end{aligned} \tag{4}$$

In this expression we set

$$\mathcal{C}_r(M) = \{(z_1,\ \ldots,\ z_r)\in M^r\ ;\ z_i\neq z_j\ (\forall i\neq j)\ \}/S_r \quad , \tag{5}$$

[3]The meaning of "chiral" is given in [3, 4].

where S_r acts as the permutations of $z_1, \ \ldots, \ z_r$, and so we can explicitely write down its Euler number;

$$\chi(\mathcal{C}_r(M)) = \frac{(2-2p)(2-2p-1)\cdots(2-2p-r+1)}{r!}. \tag{6}$$

K_v is the number of cycles in the cycle decompositon of $v \in S_n$, $\delta(\cdots)$ means the delta function on $\mathrm{C}[S_n]$ (the group algebra ove the symmetric group S_n) and $T_2 \subset S_n$ denotes the conjugacy class of the transpositions.

We survey their main results. For this purpose we shall neglect the factor $e^{\frac{n^2}{2N^2}\lambda A}$ for the time being. Rewrite (4) as $Z^+_{YM_2}(M, SU(N)) = 1 + \sum_{n=1}^{\infty}\sum_{B=0}^{\infty}\left(\frac{1}{N}\right)^{2h-2} e^{-\frac{1}{2}n\lambda A} Z^+(h,n,\lambda A)$ with the relation $2h-2 = n(2p-2)+B$ (Riemann-Hurwitz). We then find that $Z^+(h,n,\lambda A)$ is a summation of some topological invariants of the n-fold branched covers $f : \Sigma \longrightarrow M$ ($h =$ genus of Σ), or equivalently, of the holomorphic mappings with degree n. For the case of $\lambda A = 0$ they proved a remarkable fact;

$$Z^+(h,n,0) = \chi_{\mathrm{orb}}(\mathcal{F}_{h,n}), \tag{7}$$

namely, $Z^+(h,n,0)$ is equal to the orbifold Euler number of the moduli space of branched cover. Based on this fact they constructed a model of topological string such that it has $Z^+(h,n,0)$ as its partition function. They also conjectured and partially proved that the case of $\lambda A \neq 0$ is recovered by including the perturbation of the "area operator" $\sim -\frac{\lambda}{2}\int_{\Sigma} f^*\omega$, where ω is the volume form of M, with carefully treating the contact terms in the similar manner as those of topological gravity [11].

Let us return to the formula (4). In the summation in the formula (4) the factors $\frac{1}{k!}(-\lambda A)^k$ and $\chi(\mathcal{C}_L(M))$ correspond to the integrals over the continuous moduli. On the other hand, each of the elements of S_n represents the combinatorial data of the branched cover, namely, the summation over the elements of S_n can be translated into the summation over the homotopy classes of branched covers. The delta function factor $\frac{1}{n!}\delta(v_1\cdots)$ imposes the consistency condition to reconstruct the covering Riemann surface. The elements $v_1, \ldots, v_L, \ p_1, \ldots, p_k \ \in S_n$ respectively correspond to the branch points, say, $V_1, \ldots, V_L$, $P_1, \ldots, P_k$ on M. Their correct contributions to the Euler number of the world sheet Σ are equal to $n - K_{v_i}$ for V_i ($i = 1, \ldots, L$) and 1 for each P_j ($j = 1, \ldots, k$). (This means that all P_j's are simple branch points.) The appearance of the factor $\delta_{\sum_{i=1}^{L}(n-K_{v_i})+k \ ,B}$ ensures the correct genus expansion of Σ.

We lastly comment on the role of the factor $e^{\frac{n^2}{2N^2}\lambda A}$ which we neglected above. A nice interpretation of this factor is presented in [3], namely, the "infinitesimal tubes and handles". But we have not succeeded in properly treating these objects and their generalizations for gYM_2 in the framework of the CMR's topological string. Fortunately, the contributions of these terms are absent for the case of $G = U(N)$. Hence we shall focus on the $U(N)$-theory in the following discussions.

3. Now let us consider the "generalized two-dimensional Yang-Mills theory". This is defined by

$$S_{gYM_2}(A_\mu, \phi) = S_{BF}(A_\mu, \phi) + \int_M dv\, V(\phi), \tag{8}$$

where the "potential" $V(\phi)$ is an arbitrary invariant polynomial. The gauge theory of this type was studied in [7, 8]. The quantization of this theory is completely parallel to YM_2 and its partition function is given by the similar formula as (3). The only difference is that the quadratic Casimir $C_2(R)$ appearing in this formula, which corresponds to the term $\sim \int dv\, \mathrm{Tr}(\phi^2)$ in (2), is replaced with some higher Casimir operator corresponding to the general potential $V(\phi)$ [7, 8].

Consider the case; $V(\phi) = g_n\, \mathrm{Tr}(\phi^n) +$ lower terms. The corresponding Casimir operator generally has the form; $g_n\, C_n(R) + \sum_{\{k_i\}} a(k_1, k_2, \ldots) \prod_r (C_r(R))^{k_r}$, where the summation of the 2nd term is taken over the set $\{k_1, k_2, \ldots\}$ such that $\sum_r r k_r < n$ and $C_r(R)$ denotes the r-th Casimir operator of $U(N)$[4];

$$C_r(R) = \sum_{i=1}^{N} m_i^r \prod_{j \neq i} \left(1 - \frac{1}{m_i - m_j}\right). \tag{9}$$

In this expression we set $m_j = n_j + N - j$ with $(n_1, \ldots, n_N)$ being the signature of R. In order to determine the coeffecients $a(k_1, k_2, \ldots)$ from $V(\phi)$ we must fix the renormalization condition explicitely (fix the defintion of the normal ordering). This is generally a very complicated procedure and beyond the scope of this article. Instead we shall simply *define* the gYM_2 by the following formula;

$$Z_{gYM_2}(M, G) = \sum_R (\dim R)^{2-2p}\, e^{-\sum_{r=1}^{n} g_r A C_r(R) + \sum_{\{k_i\}} a(k_1,k_2,\ldots) \prod_s (C_s(R))^{k_s}} \tag{10}$$

[4]Our definitions of the higher Casimirs are slitely different from those in [8]. This difference reduces to the choice of the renormalization condition.

We especially focus on the cases such that only the linear terms $\sim \sum_{r=1}^{n} g_r AC_r(R)$ are included in the "energy eigen-values" of gYM_2 and will treat the chiral sector only.

Before developing the main discussions, we need to give a few remarks: Firstly, the coupling constants g_r should be rescaled as $g_r = \dfrac{\lambda_r}{N^{r-1}}$ when taking the large N-limit. We will observe that these rescalings are necessary for the correct genus expansions. We must also remark the following fact; the $U(N)$-theory has an extra degrees of freedom, the "$U(1)$-charge", compared with the $SU(N)$-theory. The irreps of $U(N)$ are classified by the $U(1)$-charge Q and the irreps of $SU(N)$-subgroup (Yang tableaus). Here we shall only consider the sector with $Q = 0$ in order to make things easy. This has the similar structure as the $SU(N)$-theory, but easier to treat, since the higher Casimirs of $U(N)$ is much simpler than those of $SU(N)$. As has been already pointed out, the $SU(N)$-theory includes some extra terms which are the natural generalizations of the infinitesimal tubes and handles in the case of YM_2. We would like to argue on the sectors of non-zero Q and on the $SU(N)$-theory elsewhere.

Now, the question we want to solve is as follows: What is the topological string action corresponding to the model (10)? In other words, what is the perturbation terms to the CMR's string action which recover the contributions of the higher casimirs $\sim \sum_{r=1}^{n} g_r AC_r(R)$? To search for the solution of this problem let us proceed along the same line from (3) to (4). So, we need to clarify the N-dependence of $C_r(R)$. Set $R \in Y_n$ (the set of Young tableaus with n-boxes) and assume that $n << N$, since we are considering the chiral sector only. It is convenient to expand formally $C_r(R)$ with respect to N;

$$\frac{1}{N^{r-1}} C_r(R) = \sum_{l=1}^{r} \begin{pmatrix} r-1 \\ l-1 \end{pmatrix} \frac{1}{N^{l-1}} \Xi_l(R). \tag{11}$$

Here the coefficients $\Xi_l(R)$ are defined not to depend on N. One remarkable fact is that $\Xi_l(R)$'s so defined are actually independent of the value of r. The binomial coefficients $\begin{pmatrix} r-1 \\ l-1 \end{pmatrix}$ are suitably chosen to assure of this property. In the similar manner as [3, 8] we can represent $C_r(R)$, or rather $\Xi_l(R)$, in terms of the language of symmetric group. After

some calculations we can obtain

$$
\begin{aligned}
\Xi_1(R) &= n \\
\Xi_2(R) &= 2\frac{\chi_R(P_2)}{d_R} \\
\Xi_3(R) &= 3\frac{\chi_R(P_3)}{d_R} + n(n-1) \\
\Xi_4(R) &= 4\frac{\chi_R(P_4)}{d_R} + (6n-10)\frac{\chi_R(P_2)}{d_R} \\
\Xi_5(R) &= 5\frac{\chi_R(P_5)}{d_R} + 3(4n-7)\frac{\chi_R(P_3)}{d_R} \\
&\quad + 16\frac{\chi_R(P_{2,2})}{d_R} + 2n(n-1)(n-2) + n(n-1) \\
\cdots
\end{aligned}
\tag{12}
$$

In these expressions $\chi_R(\cdots)$, d_R mean the character and the dimension as the irrep of S_n.[5] $P_{n_1,n_2,\ldots}(\in \mathrm{C}[S_n])$ denotes the sum of all the elements of S_n belonging to the conjugacy class represented by $\{n_1, n_2, \ldots\} \in Y_n$, and we employ some abbreviations, say, P_2 truly means $P_{2,\underbrace{1,\ldots,1}_{n-2 \text{ times}}} \equiv$ sum of all transpositions. It may be convenient to define the elements $\Xi_l^{(n)} \in \mathrm{Center}(\mathrm{C}[S_n])$ such that $\Xi_l(R) = \rho_R(\Xi_l^{(n)}) \equiv \dfrac{\chi_R(\Xi_l^{(n)})}{d_R}$ where $\rho_R(\cdots)$ means the representation matrix. For example, $\Xi_2^{(n)} = 2P_2$, $\Xi_3^{(n)} = 3P_3 + n(n-1)$, $\ldots$ and so on.

In this way we have succeeded in translating the language of $U(N)$ into that of S_n. It is easy to write down the similar formula as (4) for gYM_2 from the formulas (12). We should recall that this procedure is a cornerstone of the stringy interpretations of YM_2 [3, 4]. So, we can already give the similar interpretations of gYM_2 in the *qualitative* level. In fact these subjects are carefully discussed in [8]. To solve our question, however, we need more *quantitative* informations. We shall restart with the following identity on $\mathrm{C}[S_n]$;

$$
\Xi_l^{(n)} = \sum_{a=1}^{n} \sum_{b_1,\ldots,b_{l-1}\neq a} p_{ab_1}\cdots p_{ab_{l-1}}, \tag{13}
$$

where $p_{ab}(\equiv (ab)\ )$ denotes the transposition acting on the a-th and b-th elements. This can be proved by simple observations about the definition of $\Xi_l^{(n)}$. This identity (13) includes the sufficient informations we want and leads to a manifest interpretation; $\Xi_l^{(n)}$ *represents the degeneration of $l-1$ simple ramification points.* Here we should notice that the factors

[5] Acoding to the convention of [3], we often use the letter R either as the meaning of the irrep of $U(N)$ ($SU(N)$) or the meaning of irrep of S_n when they are expressed by the same Young tableau. It may be somewhat confusing, but we dare to do so to avoid the notational complexities.

$(1/N)^{l-1}$ of $\Xi_l^{(n)}$ appearing in the identity (11) completely match with this interpretation. In fact the existence of $l-1$ simple ramification points adds $-(l-1)$ to the Euler number of the world sheet and therefore these factors give the correct genus expansion. If we adopt the rescaling $g_r = \dfrac{\lambda_r}{N^{r-1}}$, which we mentioned before, all the terms $\Xi_l^{(n)}$ will appear in our equations with the suitable factors $(1/N)^{l-1}$. It is an interesting feature that the N-dependences of all the coupling constants are decided from the appropriate stringy interpretaion. It might be analogous to the situation of the doubly scaled matrix models [13].

To treat our problem more concretely we must prepare some mathematical objects. Let $\mathcal{F}_{h,n,s}$ be the moduli space of the n-fold branched cover over M with genus h and s-punctures, in other words, the moduli space of toplogical string whose target space is M (n is usually called the "instanton number" or the "winding number" in the language of string theory);

$$\begin{aligned}\mathcal{F}_{h,n,s} \overset{\text{def}}{=} \{ (\Sigma,\ f,\ x_1,\ \ldots,\ x_s) \ :\ f\ :\ \Sigma \longrightarrow M\ n\text{-fold branched cover}, \\ x_1, \ldots, x_s \in \Sigma\ \}/\mathrm{Diff}\ ,\end{aligned} \tag{14}$$

More precisely we must designate a suitable compactification rule. We shall here adopt the compactification appearing in [4] with respect to the collisions of ramification points, and the stable compactification with respect to the collisions of punctures. We also use the abbreviated notation $\mathcal{F}_{h,n}$, which we used in the previous discussions, instead of $\mathcal{F}_{h,n,0}$.

Consider the one-puncture case $\mathcal{F}_{h,n,1}$. Roughly speaking, $\mathcal{F}_{h,n,1} \sim \mathcal{F}_{h,n} \times \Sigma$, so we can obtain the fibration;

$$\pi\ :\ \mathcal{F}_{h,n,1} \longrightarrow \mathcal{F}_{h,n}, \tag{15}$$

such that the fiber $\pi^{-1}([\Sigma, f]) \cong \Sigma$ describes the position of puncture.

For a fixed branched cover $f\ :\ \Sigma \to M$, we define the "ramification divisor" of Σ by

$$R_{\Sigma,f} \overset{\text{def}}{=} \sum_{p\in\Sigma} (e(p)-1)p, \tag{16}$$

where $e(p)$ is the ramification index of $p \in \Sigma$. (The summation is well-defind since $e(p) = 1$ for the unramified points.) The divisor $R_{\Sigma,f} \subset \Sigma$ naturally induces a divisor $\mathcal{R}$ of $\mathcal{F}_{h,n,1}$ defined by;

$$\mathcal{R} \cap \pi^{-1}([\Sigma, f]) = [R_{\Sigma,f}], \tag{17}$$

where $[\cdots]$ means the isomorphism class of divisor defined by the diffeomorphisms. Heuristically, $\mathcal{R}$ is nothing but the subvariety of $\mathcal{F}_{h,n,1}$ composed of the configurations such that the puncture collides with some ramification points. Let us denote the holomorphic line bundle

corresponding to $\mathcal{R}$ by $\mathcal{L} \rightarrow \mathcal{F}_{h,n,1}$. More explicitely $\mathcal{L}$ is the line bundle whose fiber is given by

$$\mathcal{L}_{[\Sigma,f,x]} = K_\Sigma \otimes f^*(TM)|_x, \tag{18}$$

where K_Σ means the canonical bundle of Σ, TM means the holomorphic tangent bundle of M. The simplest (somewhat heuristic) explanation that the line bundle $\mathcal{L}$ having this fiber indeed corresponds to the divisor $\mathcal{R}$ is as follows: Consider the holomorphic section

$$s \; : \; [\Sigma, f, x] \in \mathcal{F}_{h,n,1} \;\longmapsto\; \frac{df}{dz}(x) \in K_\Sigma \otimes f^*(TM)|_x. \tag{19}$$

Clearly the zero-locus of this section coincides with the subset of configurations such that the puncture x coincides with a ramification point, that is, $\mathcal{R}$ itself.

Let us return to the main discussions. Our goal is to construct the suitable observables reproducing the higher Casimir terms $\sim \sum_{r=1}^{n} g_r A C_r(R)$ in (10) in the framework of the CMR's string theory. We first introduce the "area-operator";

$$\mathcal{A} \stackrel{\text{def}}{=} \int f^*\omega, \tag{20}$$

where ω means the volume form on M. As is well-known [11, 12], we can equivalently use the 0-form component folded with the puncture operator P;

$$\mathcal{A} = \omega_{ij}(f(x))\, \chi^i \chi^j \, P. \tag{21}$$

where χ^i denotes the usual ghost field of the topological σ-model. Consider the operator of the form $(c_1(\mathcal{L}))^l \, \mathcal{A}$. It is analogous to the gravitational descendants of the theory of topological gravity [12, 11]. In that case one considered the line bundle whose fiber is

$$\mathcal{L}^{\text{top grav}}|_{[\Sigma,f,x]} = K_\Sigma|_x, \tag{22}$$

instead of (18). Therefore it may be reasonable to define these objects as the *deformed* gravitational descendants of the area operator. Let us express the form of the operator $c_1(\mathcal{L})$ more explicitely. Recalling the structure of fiber of $\mathcal{L}$ (18), we can immediately find that

$$c_1(\mathcal{L}) = c_1(\mathcal{L}^{\text{top grav}}) + R_{ij}(f(x))\, \chi^i \chi^j. \tag{23}$$

The first term in the R.H.S corresponds to the usual gravitational descendant (the operator often written as "γ^0" in the several papers of topological gravity), and the second term means nothing but the curvature two form on the target space M.

In order to clarify the geometrical meanings of these operators let us consider the Poincare dual of them. First we notice that

$$\begin{aligned}\text{Poincare dual of } \mathcal{A} &\cong \mathcal{P}_y \\ &\equiv \{\text{configurations such that the puncture} \\ &\qquad\qquad \text{is mapped to } y \in M\ \} \quad ,\end{aligned} \tag{24}$$

with a some fixed point $y \in M$. It is also convenient to introduce

$$\begin{aligned}\mathcal{R} &= \mathcal{R}_1 \supset \mathcal{R}_2 \supset \mathcal{R}_3 \supset \cdots \\ \mathcal{R}_l &= \{\text{configurations such that the puncture collides with the point} \\ &\qquad\qquad \text{which is a degeneration of } l \text{ simple ramification points }\}.\end{aligned} \tag{25}$$

Clearly $\operatorname{codim}\mathcal{R}_l = l$ in $\mathcal{F}_{h,n,1}$. Recalling that $c_1(\mathcal{L})$ is the Poincare dual of the divisor $\mathcal{R}$, we now obtain that

$$\text{Poincare dual of } (c_1(\mathcal{L}))^l\,\mathcal{A} \cong \mathcal{R}_l \cap \mathcal{P}_y. \tag{26}$$

This represents the geometrical meaning of $(c_1(\mathcal{L}))^l\,\mathcal{A}$ and suggests the relation with $\Xi^{(n)}_{l+1}$ introduced above.

There is still one thing that we should remark: Since the CMR's topological string theory has the vanishing ghost number anomaly, it is covenient to treat only the operators with the vanishing ghost numbers. Therefore, we shall rather consider the operator $\mathcal{C}^l\,\mathcal{A}$ with the definition

$$\mathcal{C} \stackrel{\text{def}}{=} c_1(\mathcal{L})\,(\hat{\boldsymbol{A}}, \Pi_0\,\hat{\boldsymbol{A}}), \tag{27}$$

instead of $(c_1(\mathcal{L}))^l\,\mathcal{A}$ itself. Here $\hat{\boldsymbol{A}}$ denotes the "co-anti ghosts" [4], which is, in some sense, the dual of the ghost fields and has the same spin contents as those of the ghosts. Π_0 is the projector onto the space of zero-modes. This is actually a BRST-invariant object. The projector Π_0 is needed to assure this invariance for the factor $(\hat{\boldsymbol{A}}, \Pi_0\,\hat{\boldsymbol{A}})$, of which role is merely to cancel the ghost number of $c_1(\mathcal{L})$.

We are now in a position to demonstrate the main results of this article. Let us write the correlator of the CMR's topological string defined on the world sheet Σ with genus h as

$$\langle \cdots \rangle_h \equiv \int D(\boldsymbol{F}, \hat{\boldsymbol{F}}, \cdots\ )\ \cdots\ e^{-I_{\mathrm{CMR}}(\boldsymbol{F}, \hat{\boldsymbol{F}}, \cdots\ )}, \tag{28}$$

and similarly express the contribution from the n-instanton sector by $\langle \cdots \rangle_{h,n}$. One of the main results of [4] can be written as

$$\langle\, 1\, \rangle_{h,n} = \chi_{\mathrm{orb}}(\mathcal{F}_{h,n}). \tag{29}$$

First we should notice the following relation;

$$\langle \mathcal{A} \rangle_h = \sum_{n=1}^{\infty} nA\langle 1\rangle_{h,n} + \langle \mathcal{C}\,\mathcal{A}\rangle_h. \tag{30}$$

The second term in the R.H.S represents the contact terms between the area operator $\mathcal{A}$ and the ramification points which are discussed in [4]. We may also rewrite it as follows;

$$\begin{aligned} \langle (1-\mathcal{C})\,\mathcal{A}\rangle_h &= \sum_{n=1}^{\infty} nA\langle 1\rangle_{h,n} \\ &= A\sum_{n=1}^{\infty}\sum_{L=0}^{\infty} \chi(\mathcal{C}_L(M)) \sum_{v_1,\ \dots,\ v_L\in S_n\backslash\{1\}} \ \sum_{s_1,\ t_1,\ \dots,\ s_p,\ t_p\in S_n} \\ &\quad \times \frac{1}{n!}\delta(\Xi_1^{(n)}\, v_1\cdots v_L \prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1})\,\delta_{\sum_{i=1}^{L}(n-K_{v_i}),B(h,n)}\ , \end{aligned} \tag{31}$$

where $B(h,n)$ is defined by $2h-2 = n(2p-2)+B(h,n)$. Remembering the formulas (13), (26) and the definition of $\mathcal{R}_l$, we can further obtain the analogous identity for the deformed gravitational descendants;

$$\begin{aligned} \langle\,(1-\mathcal{C})\,\mathcal{C}^l\,\mathcal{A}\,\rangle_h &= A\sum_{n=1}^{\infty}\sum_{L=0}^{\infty} \chi(\mathcal{C}_L(M)) \sum_{v_1,\ \dots,\ v_L\in S_n\backslash\{1\}} \ \sum_{s_1,\ t_1,\ \dots,\ s_p,\ t_p\in S_n} \\ &\quad \times \frac{1}{n!}\delta(\Xi_{l+1}^{(n)}\, v_1\cdots v_L \prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1})\,\delta_{\sum_{i=1}^{L}(n-K_{v_i})+l\,,B(h,n)}. \end{aligned} \tag{32}$$

The factor $(1-\mathcal{C})$ suppresses again the extra contributions from the contact terms. If we incorpolate the suitable factors $(1/N)^l$ for $\Xi_{l+1}^{(n)}$, we can find out the simple correspondence;

$$(1-\mathcal{C})\,\mathcal{C}^l\,\mathcal{A}\ \longleftrightarrow\ A\left(\frac{1}{N}\right)^l \Xi_{l+1}^{(n)}. \tag{33}$$

Recalling the relation of the Casimir operators and $\Xi_l^{(n)}$ (11), we finally obtain

$$\begin{aligned} (1-\mathcal{C})(1+\mathcal{C})^l\,\mathcal{A} &\equiv \sum_{r=0}^{l}\begin{pmatrix} l \\ r\end{pmatrix}(1-\mathcal{C})\,\mathcal{C}^r\,\mathcal{A} \\ &\longleftrightarrow\ A\sum_{r=0}^{l}\begin{pmatrix} l \\ r\end{pmatrix}\left(\frac{1}{N}\right)^r \Xi_{r+1}^{(n)}\ \longleftrightarrow\ A\left(\frac{1}{N}\right)^l\ C_{l+1}(R). \end{aligned} \tag{34}$$

In this way we have arrived at the solution of our main question: *gYM_2 including the higher Casimir terms $\sim A\sum_{r=1}^{m}\frac{\lambda_r}{N^{r-1}}C_r(R)$ can be recovered by the CMR's topological string with the perturbation term*

$$\sum_{r=1}^{m}\lambda_r\int(1-\mathcal{C})(1+\mathcal{C})^{r-1}\,\mathcal{A}. \tag{35}$$

To close our discussions let us present a few comments: Firstly, notice that the area-area contact terms vanish [4]. Hence the results (30) (31) (32) can be exponentiated with no difficulty and one can obtain the correct polynomials of A. This immediately leads to our final results.

Secondly, let us consider the special case for $C_2(R)$ - the case of usual YM_2. According to the above rule we obtain as the appropriate perturbation term $\sim\ \lambda_2 \int (1-\mathcal{C}^2)\,\mathcal{A}$. This does not coincide with the CMR's original conjecture; $\sim\ \lambda_2 \int \mathcal{A}$. But their arguments are those about only the simple Hurwitz space (= the subspace of $\mathcal{F}_{h,n}$ composed of the configurations such that all the ramification points are simple). Hence our result is compatible with theirs, since the correction term $-\mathcal{C}^2\,\mathcal{A}$ does not contribute to the simple Hurwitz space.

4. In this article we investigated the two-dimensional generalized Yang-Mills theories from the point of view of the topological string theory. We have uncovered the geometrical meaning of the higher Casimirs by recasting them, namely, by introducing the elements $\Xi_l^{(n)}$ of $\mathrm{C}[S_n]$. Our most remarkable success is to present the expressions of the perturbation terms appropriate to describe the models of gYM_2 including rather general higher Casimirs. But it is *not* completely general. We have not yet succeeded in treating more general forms of higher Casimirs including the non-linear terms. We also have not succeeded in treating the sector with non-zero $U(1)$-charges, and working with the $SU(N)$-theory. They are the open problems to be resolved, and I think, these problems might deeply relate with one another. In order to overcome these problems it will be necessary to develop our theory so that we can work with the terms of "infinitesimal tubes and hundles", which we neglected.

Another open problem is of course to extend our results to the non-chiral case. To this aim it may be more helpful to work in the framework of Hořava's theory ("topological rigid string") [9] than that of CMR's theory. His string model possesses the moduli space of the minimal area maps rather than the (anti-)holomorphic maps. Hence the non-chiral sector is naturally incorporated from the begining.

The problem that I think also interesting is as follows: The CMR's studies disclosed the deep relation between YM_2 (and gYM_2) and the Euler characteristic of moduli space. On the other hand, the matrix models of the Kontsevich-Penner type [14] give the analogous results. It is well-known that the matrix models of this type naturally give the simplicial decompositions of the moduli space by the Feynmann diagrams. It may be meaningful if we can develop the similar diagramatic arguments for YM_2 (and gYM_2).

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